Graduate Macroeconomic Theory (ECN 200E)
Discussion Section Week 8

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Forward vs. Backward Iteration

Solve Forward or Backward?

Given a linear difference equation in x_t:

$$x_t = \lambda^{-1}\mathbb{E}_t x_{t+1} + f_t$$

When do we solve it forward? When backward?

- If $|\lambda| > 1$: forward iteration kills the explosive term $\Rightarrow$ **iterate forward**
- If $|\lambda| < 1$: backward iteration kills the explosive term $\Rightarrow$ **iterate backward**
- Picking the wrong direction $\Rightarrow$ explosive (non-stationary) solution

You always want to solve in the direction where the **coefficient on the iterated term shrinks to zero**.

- Forward: $\lambda^{-T} \to 0$ iff $|\lambda| > 1$
- Backward: $\lambda^{T} \to 0$ iff $|\lambda| < 1$

Case 1: $|\lambda| > 1$ — Unstable root

Generic first-order expectational difference equation:

$$\mathbb{E}_t x_{t+1} = \lambda x_t + g_t$$

Case 1: $|\lambda| > 1$ — Unstable root **Rearrange to iterate forward:**

$$x_t = \lambda^{-1}\mathbb{E}_t x_{t+1} - \lambda^{-1} g_t$$

Iterate T times:

$$x_t = -\sum_{k=0}^{T} \lambda^{-(k+1)}\mathbb{E}_t g_{t+k} + \lambda^{-(T+1)}\mathbb{E}_t x_{t+T+1}$$

Since $|\lambda| > 1$: terminal term $\to 0$.

Unique stable solution:

$$x_t = -\sum_{k=0}^{\infty} \lambda^{-(k+1)}\mathbb{E}_t g_{t+k}$$

Case 2: $|\lambda| < 1$ — Stable root

Case 2: $|\lambda| < 1$ — Stable root **Rearrange to iterate backward:**

$$x_{t+1} = \lambda x_t + g_t + \xi_{t+1}, \qquad \mathbb{E}_t \xi_{t+1} = 0.$$

Iterate backwards from initial x_0:

$$x_t = \lambda^t x_0 + \sum_{j=0}^{t-1} \lambda^{t-1-j} g_j + \sum_{j=0}^{t-1} \lambda^{t-1-j} \xi_{j+1}.$$

Since $|\lambda| < 1$: **any** mean-zero shock ξ works.

Multiple solutions $\Rightarrow$ indeterminate!

Example: Taylor rule with $\phi_\pi < 1$;

Money Targeting in the Classical Monetary Model[1]

[1] Reference: Galí (2020), Ch. 2

Setup and Key Ingredients

Policy regime: The central bank sets an exogenous path $\{m_t\}$ for the money supply. We need two equilibrium conditions from the model:

(1) Money demand (from MIU, log-linearized):

$$m_t - p_t = y_t - \eta\, i_t, \qquad \eta \geq 0$$

$\eta =$ interest semi-elasticity of money demand.

(2) Fisher equation (exact log-linear relationship):

$$i_t = \mathbb{E}_t\{p_{t+1} - p_t\} + r_t = \mathbb{E}_t\{\pi_{t+1}\} + r_t$$

Step 1: Eliminating the Nominal Interest Rate

From Fisher (21):

$$i_t = \mathbb{E}_t\{p_{t+1}\} - p_t + r_t$$

Substitute into money demand (10):

$$m_t - p_t = y_t - \eta\big(\mathbb{E}_t\{p_{t+1}\} - p_t + r_t\big)$$

Solve for p_t:

$$(1+\eta)\, p_t = \eta\, \mathbb{E}_t\{p_{t+1}\} + m_t - y_t + \eta\, r_t$$

Difference equation for the price level

$$p_t = \left(\frac{\eta}{1+\eta}\right) \mathbb{E}_t\{p_{t+1}\} + \left(\frac{1}{1+\eta}\right) m_t + u_t$$

where $u_t \equiv (1+\eta)^{-1}(\eta\, r_t - y_t)$ evolves *independently* of $\{m_t\}$.

Step 2: Stability Check

$$p_t = \underbrace{\left(\frac{\eta}{1+\eta}\right)}_{\equiv \beta_p} \mathbb{E}_t\{p_{t+1}\} + \frac{1}{1+\eta} m_t + u_t$$

Key observation: For any $\eta > 0$:

$$0 \; < \; \frac{\eta}{1+\eta} \; < \; 1$$

Rewrite as a first-order difference equation with leading term:

$$\mathbb{E}_t\{p_{t+1}\} = \frac{1+\eta}{\eta} p_t - \frac{1}{\eta} m_t - \frac{1+\eta}{\eta} u_t$$

The coefficient on p_t is $\frac{1+\eta}{\eta} > 1$: **unstable root**.

$\Rightarrow$ We iterate **forward** to find the unique bounded solution.

$\Rightarrow$ Terminal condition: $\left(\frac{\eta}{1+\eta}\right)^T \mathbb{E}_t\{p_{t+T}\} \to 0$ as $T \to \infty$.

Step 3: Forward Iteration

Start from:

$$p_t = \frac{\eta}{1+\eta}\mathbb{E}_t\{p_{t+1}\} + \frac{1}{1+\eta}m_t + u_t$$

After T forward iterations:

$$p_t = \left(\frac{\eta}{1+\eta}\right)^{T+1}\mathbb{E}_t\{p_{t+T+1}\} + \frac{1}{1+\eta}\sum_{k=0}^{T}\left(\frac{\eta}{1+\eta}\right)^k\mathbb{E}_t\{m_{t+k}\} + \sum_{k=0}^{T}\left(\frac{\eta}{1+\eta}\right)^k\mathbb{E}_t\{u_{t+k}\}$$

Take $T \to \infty$: Terminal condition vanishes since $\frac{\eta}{1+\eta} < 1$.

Forward solution for the price level

$$p_t = \frac{1}{1+\eta}\sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k\mathbb{E}_t\{m_{t+k}\} + u_t'$$

where $u_t' \equiv \sum_{k=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^k\mathbb{E}_t\{u_{t+k}\}$ is independent of monetary policy.

Step 4: Rewriting in Terms of Money Growth

Trick: For any $k \geq 1$, write $m_{t+k} = m_t + \sum_{j=1}^{k} \Delta m_{t+j}$.

$$p_t = \frac{1}{1+\eta} \sum_{k=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k \left(m_t + \sum_{j=1}^{k} \mathbb{E}_t\{\Delta m_{t+j}\} \right) + u_t'$$

$$p_t = m_t \cdot \underbrace{\frac{1}{1+\eta} \sum_{k=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k}_{=1} + \frac{1}{1+\eta} \sum_{k=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k \sum_{j=1}^{k} \mathbb{E}_t\{\Delta m_{t+j}\} + u_t'$$

Galí equation (25)

$$\boxed{p_t = m_t + \sum_{k=1}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k \mathbb{E}_t\{\Delta m_{t+k}\} + u_t'}$$

The price level = current money supply + a **discounted sum of expected future money growth**.

Step 5: The AR(1) Example

Special case: Money growth follows an AR(1):

$$\Delta m_t = \rho_m \, \Delta m_{t-1} + \varepsilon_t^m, \qquad \rho_m \in [0, 1)$$

Simplifying assumption: No real shocks $\Rightarrow r_t = y_t = 0$ for all t, so $u_t' = 0$.

From the AR(1) process, expected future money growth:

$$\mathbb{E}_t\{\Delta m_{t+k}\} = \rho_m^k \, \Delta m_t \qquad \text{for all } k \geq 1$$

Substitute into Galí eq. (25):

$$p_t = m_t + \sum_{k=1}^{\infty} \left(\frac{\eta}{1+\eta} \right)^k \rho_m^k \, \Delta m_t = m_t + \Delta m_t \sum_{k=1}^{\infty} \left(\frac{\eta \rho_m}{1+\eta} \right)^k$$

Step 5: The AR(1) Example

Closed-form solution for the price level

$$p_t = m_t + \frac{\eta\rho_m}{1+\eta(1-\rho_m)}\Delta m_t$$

Nominal interest rate (from money demand, with $y_t = 0$):

$$i_t = \eta^{-1}(m_t - p_t)\cdot(-1) = \eta^{-1}\frac{\eta\rho_m}{1+\eta(1-\rho_m)}\Delta m_t$$

Nominal interest rate under AR(1) money growth

$$i_t = \frac{\rho_m}{1+\eta(1-\rho_m)}\Delta m_t$$

Two striking predictions — both counterfactual

1. p_t responds **more than one-for-one** with m_t (when $\rho_m > 0$)
2. i_t **rises** in response to a money supply expansion (no liquidity effect)

Intuition: Why Does p_t Move More Than One-for-One?

The formula:

$$p_t = m_t + \underbrace{\frac{\eta \rho_m}{1 + \eta(1 - \rho_m)}}_{\equiv \Phi > 0} \Delta m_t, \qquad \Phi > 0 \textit{ iff } \rho_m > 0$$

Suppose the CB increases m_t by 1 unit today, and $\rho_m > 0$.

Expected future money supply also rises: $\mathbb{E}_t\{m_{t+k}\} \uparrow$ for all $k \geq 1$.

Forward-looking logic

- From eq. (25): p_t reflects not just m_t but also all future $\mathbb{E}_t\{m_{t+k}\}$
- If $\rho_m > 0$: a rise in Δm_t signals persistently higher future money growth
- Agents expect higher future prices $\Rightarrow$ they want to spend money *now*
- Excess demand drives up p_t **beyond** the direct effect of the current m_t increase

Intuition: Why Does p_t Move More Than One-for-One?

The coefficient Φ measures this forward-looking amplification:

$$\Phi = \frac{\eta\rho_m}{1+\eta(1-\rho_m)} \begin{cases} = 0 & \text{if } \rho_m = 0 \text{ (i.i.d.: no amplification)} \\ > 0 & \text{if } \rho_m > 0 \text{ (persistent: amplification)} \\ \to \eta & \text{as } \rho_m \to 1 \text{ (permanent: max amplification)} \end{cases}$$

Intuition: Why Does the Interest Rate Rise? (No Liquidity Effect)

The formula:

$$i_t = \frac{\rho_m}{1 + \eta(1 - \rho_m)} \Delta m_t$$

A positive money supply shock ($\Delta m_t > 0$) raises i_t when $\rho_m > 0$.

Fisher equation logic

$$i_t = \underbrace{r_t}_{=0} + \mathbb{E}_t\{\pi_{t+1}\}$$

- Higher persistent money growth $\Rightarrow$ higher expected inflation $\mathbb{E}_t\{\pi_{t+1}\}$
- Fisher equation: higher expected inflation raises i_t one-for-one (r_t not affected)
- The **inflation expectations channel** dominates over any liquidity channel

Intuition: Why Does the Interest Rate Rise? (No Liquidity Effect)

Contrast with empirical evidence

- Empirically: A monetary expansion lowers i_t in the short run (**liquidity effect**) and raises p_t only *sluggishly*.
- This model predicts the opposite: instantaneous price jump + rising interest rates.
- Why the failure? Perfectly flexible prices $\Rightarrow$ no room for real effects.
- The fix: **nominal rigidities** (New Keynesian model, Ch. 3).

Example Question: Inertial Taylor Rule[2]

[2] Final Exam 2023 WQ

Example Question: Inertial Taylor rule

Nominal block: Fisher equation + interest-rate rule.

Fisher:

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}, \qquad \hat{r}_t = -\hat{e}_t, \quad \hat{e}_t \text{ i.i.d.}$$

Inertial Taylor rule:

$$\hat{i}_t = \phi_i \hat{i}_{t-1} + (1 - \phi_i)\phi_\pi \hat{\pi}_t, \qquad 0 < \phi_i < 1.$$

Example Question: Inertial Taylor rule

Questions:
(a): Iterate (backward) to show equivalently

$$\hat{i}_t = \phi_\pi \bar{\pi}_t, \qquad \bar{\pi}_t = (1-\phi_i)\sum_{k=0}^{\infty} \phi_i^k \hat{\pi}_{t-k}.$$

(b) Show: if $\phi_\pi > 1$, there exists a *unique stable* solution for $\bar{\pi}_t$. Note that "average" inflation can be expressed as: $\bar{\pi}_t = \phi_i \bar{\pi}_{t-1} + (1-\phi_i)\,\hat{\pi}_t$

(c) Assume $\phi_\pi > 1$. Solve explicitly for $\bar{\pi}_t$ (as a function of shocks).

(d) One-period TFP shock: $\hat{e}_t \neq 0$, $\hat{e}_{t+k} = 0$ for $k \geq 1$. Find the path of actual inflation $\hat{\pi}_t$ over time; discuss $t, t+1, t+2$.