Graduate Macroeconomic Theory (ECN 200E)
Week 7: Money in Utility (MIU) Model

Jinho Kim

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1 Motivation

Why MIU?

- Generates an explicit **money demand condition** from household optimization.
- Tractable in DSGE / New Keynesian models:
  - smooth first-order conditions
  - easy log-linearization
  - direct link between real balances and nominal interest rate
- Convenient reduced-form way to capture **liquidity services of money**.

Historical benchmark

- Earlier macro models often used **reduced-form money demand** (e.g. LM-type relations).
- MIU embeds money demand inside a standard intertemporal optimization problem.

Alternative: Cash-in-Advance (CIA)

- Transaction constraint:

$$P_t C_t \leq M_t \quad \text{or commonly } P_t C_t \leq M_{t-1}$$

- Interpretation: consumption purchases require cash.
- Provides a more literal **transactions-based microfoundation**.

MIU vs CIA

- CIA:
  - inequality constraint
  - may generate binding vs slack regimes
  - kinks and solution complications
- MIU:
  - smooth Euler-type money demand
  - easier solution and estimation
  - widely used in linearized DSGE models

2 Household Problem

The representative household maximizes expected lifetime utility:

$$\max_{\{C_t, M_t, N_t, B_t\}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, \frac{M_t}{P_t}, N_t\right)$$

subject to the nominal budget constraint

$$P_t C_t + Q_t B_t + M_t \leq B_{t-1} + M_{t-1} + W_t N_t + \Pi_t$$

where Q_t is the price of a one-period nominal bond.

Nominal bond

- One-period riskless nominal bond.
- Pays one unit of money next period.

$$Q_t \equiv e^{-i_t} \approx \frac{1}{1+i_t}$$

Money vs bonds

- Both are **nominal assets** in the household balance sheet.
- Both provide future purchasing power (subject to inflation risk).
- Key difference:
  - Bonds earn nominal interest.
  - Money yields no interest but provides liquidity services.
- Therefore holding money involves an **opportunity cost** equal to the nominal interest rate.

3 Rewriting the Budget Constraint

Define total nominal financial wealth:

$$F_t \equiv B_t + M_t$$

Then

$$P_t C_t + Q_t(F_t - M_t) + M_t \leq F_{t-1} + W_t N_t + \Pi_t$$

Rearranging,

$$P_t C_t + Q_t F_t + (1 - Q_t) M_t \leq F_{t-1} + W_t N_t + \Pi_t$$

Divide by the price level:

$$C_t + \frac{Q_t}{P_t} F_t + (1 - Q_t)\frac{M_t}{P_t} \leq \frac{F_{t-1}}{P_t} + \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t}$$

Term $(1 - Q_t)M_t/P_t$ is the opportunity cost of holding money. Holding money instead of bonds implies forgone interest.

4 Utility Specification

Assume

$$U = \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \frac{\left(\frac{M_t}{P_t}\right)^{1-\gamma} - 1}{1-\gamma} - \frac{N_t^{1+\psi}}{1+\psi}$$

with preference shifter Z_t.

Preference shifter

- Z_t scales the overall level of period utility.
- It affects marginal utility but not intratemporal trade-offs.
- Relative importance across consumption, real balances, and labor is unchanged.
- Acts like a time-varying effective discount factor.
- Equivalently, it scales the stochastic discount factor.

5 Lagrangian and FOCs

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg[U(C_t, M_t/P_t, N_t)$$

$$+\lambda_t \left(\frac{F_{t-1}}{P_t} + \frac{W_t}{P_t} N_t + \frac{\Pi_t}{P_t} - C_t - \frac{Q_t}{P_t} F_t - (1 - Q_t) \frac{M_t}{P_t} \right) \Bigg]$$

Consumption

$$\beta^t C_t^{-\sigma} Z_t - \beta^t \lambda_t = 0$$

$$\boxed{\lambda_t = C_t^{-\sigma} Z_t}$$

Financial Assets

$$\beta^t \lambda_t (-Q_t) + E_t \left[\beta^{t+1} \lambda_{t+1} \frac{P_t}{P_{t+1}} \right] = 0$$

$$\boxed{\lambda_t Q_t = E_t \left[\beta \lambda_{t+1} \frac{P_t}{P_{t+1}} \right]}$$

Money

$$\beta^t \left(\frac{M_t}{P_t} \right)^{-\gamma} Z_t - \beta^t \lambda_t (1 - Q_t) = 0$$

$$\boxed{\left(\frac{M_t}{P_t} \right)^{-\gamma} Z_t = \lambda_t (1 - Q_t)}$$

Money demand condition:
Marginal utility of holding money = real marginal opportunity cost.

Labor

$$-\beta^t N_t^{\psi} Z_t + \beta^t \lambda_t \frac{W_t}{P_t} = 0$$

$$\boxed{N_t^{\psi} Z_t = \lambda_t \frac{W_t}{P_t}}$$

6 Log-Linear Intertemporal Euler Equation

From the bond FOC (using $\lambda_t = Z_t C_t^{-\sigma}$):

$$Q_t\, C_t^{-\sigma} Z_t = E_t\left[\beta\, C_{t+1}^{-\sigma} Z_{t+1}\, \frac{P_t}{P_{t+1}}\right].$$

Divide both sides by $C_t^{-\sigma} Z_t$ and multiply by Q_t^{-1}:

$$1 = E_t\left[\beta\, Q_t^{-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\frac{P_t}{P_{t+1}}\right].$$

Log-rate convention
We treat i_t as a continuously compounded (log) nominal rate:

$$Q_t = e^{-i_t} \quad\Rightarrow\quad Q_t^{-1} = e^{i_t}, \qquad \ln Q_t^{-1} = i_t.$$

Also define inflation (log price growth)

$$\pi_{t+1} \equiv \ln\frac{P_{t+1}}{P_t}$$

Take logs of the multiplicative term *inside* the expectation by writing it as an exponential:

$$\beta\, Q_t^{-1}\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\frac{P_t}{P_{t+1}} = \exp\left(\ln\beta + \ln Q_t^{-1} - \sigma\ln\frac{C_{t+1}}{C_t} + \ln\frac{Z_{t+1}}{Z_t} + \ln\frac{P_t}{P_{t+1}}\right).$$

Use $\ln Q_t^{-1} = i_t$ and $\ln(P_t/P_{t+1}) = -\pi_{t+1}$:

$$1 = E_t\left[\exp\left(\ln\beta + i_t - \sigma\ln\frac{C_{t+1}}{C_t} + \ln\frac{Z_{t+1}}{Z_t} - \pi_{t+1}\right)\right].$$

Define $\rho \equiv -\ln\beta$ and growth rates

$$\Delta c_{t+1} \equiv \ln C_{t+1} - \ln C_t, \qquad \Delta z_{t+1} \equiv \ln Z_{t+1} - \ln Z_t,$$

then

$$1 = E_t\left[\exp\left(-\rho + i_t - \sigma\,\Delta c_{t+1} + \Delta z_{t+1} - \pi_{t+1}\right)\right].$$

In a perfect-foresight steady state with constant inflation $\bar{\pi}$ and consumption growth γ,

$$\Delta c_{t+1} = \gamma, \qquad \pi_{t+1} = \bar{\pi}, \qquad \Delta z_{t+1} = 0.$$

Plugging into the exponent and imposing the steady-state Euler equation yields

$$0 = -\rho + \bar{i} - \sigma\gamma - \bar{\pi} \quad\Rightarrow\quad \boxed{\bar{i} = \rho + \sigma\gamma + \bar{\pi}.}$$

Economic intuition

- Higher trend growth γ implies higher expected future consumption, hence lower future marginal utility.
- To satisfy optimality, the steady-state real rate must rise: $\bar{r} = \rho + \sigma\gamma$.
- Nominal rate adds inflation: $\bar{i} = \bar{r} + \bar{\pi}$.

Let the exponent be

$$X_{t+1} \equiv -\rho + i_t - \sigma\,\Delta c_{t+1} + \Delta z_{t+1} - \pi_{t+1}.$$

The Euler equation is $1 = E_t[\exp(X_{t+1})]$.

Linearize around the steady state $\bar{X} = 0$ (since $-\rho + \bar{i} - \sigma\gamma - \bar{\pi} = 0$). Using the first-order Taylor expansion around 0:

$$\exp(X_{t+1}) \approx 1 + X_{t+1},$$

we get

$$1 \approx E_t[1 + X_{t+1}] \quad \Rightarrow \quad 0 \approx E_t[X_{t+1}].$$

Subtract the steady state and express in deviations. Define

$$\hat{i}_t = i_t - \bar{i}, \qquad \hat{\pi}_{t+1} = \pi_{t+1} - \bar{\pi}, \qquad \Delta\hat{c}_{t+1} = \Delta c_{t+1} - \gamma, \qquad \Delta\hat{z}_{t+1} = \Delta z_{t+1} - 0.$$

Then the linearized Euler condition becomes

$$0 = E_t\Big[\hat{i}_t - \sigma\,\Delta\hat{c}_{t+1} + \Delta\hat{z}_{t+1} - \hat{\pi}_{t+1}\Big].$$

Rearrange:

$$\boxed{E_t\,\Delta\hat{c}_{t+1} = \frac{1}{\sigma}E_t(\hat{i}_t - \hat{\pi}_{t+1}) + \frac{1}{\sigma}E_t\,\Delta\hat{z}_{t+1}.}$$

Deriving $\hat{c}_{t+1} \equiv \Delta c_{t+1} - \gamma$

$$\Delta\hat{c}_{t+1} = \Delta\ln C_{t+1} - \gamma = (\ln C_{t+1} - \ln C_t) - \gamma.$$

$$\text{Since } \gamma \equiv \Delta\ln\bar{C} = \ln\bar{C}_{t+1} - \ln\bar{C}_t,$$

we can subtract the steady-state growth:

$$\Delta\hat{c}_{t+1} = (\ln C_{t+1} - \ln C_t) - (\ln\bar{C}_{t+1} - \ln\bar{C}_t).$$

Rearrange terms:

$$\Delta\hat{c}_{t+1} = (\ln C_{t+1} - \ln\bar{C}_{t+1}) - (\ln C_t - \ln\bar{C}_t).$$

Using the definition $\hat{c}_t \equiv \ln C_t - \ln\bar{C}_t$:

$$\boxed{\Delta\hat{c}_{t+1} = \hat{c}_{t+1} - \hat{c}_t.}$$

7 Log-Linear Money Demand

From the money FOC:

$$\left(\frac{M_t}{P_t}\right)^{-\nu} Z_t = \lambda_t(1 - Q_t)$$

Using $\lambda_t = Z_t C_t^{-\sigma}$:

$$\left(\frac{M_t}{P_t}\right)^{-\nu} = C_t^{-\sigma}(1 - Q_t)$$

$$\frac{M_t}{P_t} = C_t^{\sigma/\nu}(1 - Q_t)^{-1/\nu}$$

Taking logs:

$$\ln \frac{M_t}{P_t} = \frac{\sigma}{\nu} \ln C_t - \frac{1}{\nu} \ln(1 - Q_t)$$

Substitute the exact linearization:

$$\ln \frac{M_t}{P_t} = \frac{\sigma}{\nu} \ln C_t - \frac{1}{\nu} \left[\ln(1 - \bar{Q}) + \frac{1}{e^{\bar{i}} - 1} \hat{i}_t \right]$$

$e^{\bar{i}} - 1 \approx \bar{i}$ **(small-rate approximation)**
For small steady-state nominal rates, use the first-order Taylor expansion around 0:

$$e^{\bar{i}} = 1 + \bar{i} + \mathcal{O}(\bar{i}^2) \quad \Rightarrow \quad e^{\bar{i}} - 1 = \bar{i} + \mathcal{O}(\bar{i}^2).$$

Hence

$$\frac{1}{e^{\bar{i}} - 1} \approx \frac{1}{\bar{i}} \qquad \text{(up to first order in } \bar{i}\text{)}.$$

Remove constants and express in deviations:

$$\boxed{\hat{m}_t - \hat{p}_t = \frac{\sigma}{\nu} \hat{c}_t - \frac{1}{\nu(e^{\bar{i}} - 1)} \hat{i}_t \quad \approx \quad \frac{\sigma}{\nu} \hat{c}_t - \frac{1}{\nu \bar{i}} \hat{i}_t}$$

Economic interpretation of log-linear money demand

$$\hat{m}_t - \hat{p}_t = \frac{\sigma}{\nu}\hat{c}_t - \eta\,\hat{i}_t$$

- σ: curvature of consumption utility (EIS $= 1/\sigma$). Larger $\sigma \Rightarrow$ money demand more responsive to consumption.
- ν: curvature of liquidity services. Larger $\nu \Rightarrow$ marginal utility of money falls faster $\Rightarrow$ smaller interest semi-elasticity.
- i_t: opportunity cost of holding money. Higher $i_t \Rightarrow$ lower real money demand.

8 Log-Linear Labor Supply Curve

The labor supply curve from the labor FOC is:

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\psi}$$

Take logs and deviations from steady state ($\hat{x}_t \equiv \ln X_t - \ln \bar{X}$):

$$\ln \frac{W_t}{P_t} = \sigma \ln C_t + \psi \ln N_t,$$

$$\boxed{\hat{w}_t - \hat{p}_t = \sigma \hat{c}_t + \psi \hat{n}_t}$$

Frisch elasticity of labor supply
Holding the marginal utility of consumption fixed (i.e. $\hat{c}_t = 0$), the elasticity of labor with respect to the real wage is

$$\varepsilon_F \equiv \frac{\partial \ln N_t}{\partial \ln(W_t/P_t)} = \frac{1}{\psi}.$$

Thus $1/\psi$ is the Frisch elasticity.