Graduate Macroeconomic Theory (ECN 200E)
Discussion Section Week 3

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1. Big Picture

Why Do We Linearize? (Big Picture)

- Most DSGE models are **nonlinear** and cannot be solved analytically
- We care about **local dynamics** around a stable long-run equilibrium
- Linearization provides a **tractable approximation** to study dynamics

Key idea

Linearization approximates a nonlinear model by a linear system *around the steady state*.

- Makes the model solvable using linear algebra
- Allows us to characterize dynamics, stability, and impulse responses

How We Solve DSGE Models by Linearization

1. Specify preferences, technology, and constraints
2. Derive optimality conditions and equilibrium equations
3. Turn off uncertainty and solve for the deterministic steady state
4. Linearize all equations around the steady state
5. Rewrite the model as a linear dynamic system
6. Apply Blanchard–Kahn conditions
7. Obtain policy functions (choices as functions of states)

Big picture

Linearization transforms a nonlinear equilibrium model into a tractable linear system that describes local dynamics.

2. Linearizing Examples

Stochastic Growth Model

$$y_t = c_t + i_t \tag{1}$$
$$k_{t+1} = (1-\delta)k_t + i_t \tag{2}$$
$$c_t^{-\sigma} = \beta E_t[c_{t+1}^{-\sigma} r_{t+1}] \tag{3}$$
$$lna_t = \rho lna_{t-1} + \epsilon_t \tag{4}$$
$$y_t = a_t k_t^{\alpha} \tag{5}$$
$$r_{t+1} = \alpha a_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \tag{6}$$

Key approximation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$\frac{dx_t}{x_t} = \frac{x_t - \bar{x}}{\bar{x}} \equiv \hat{x}_t \quad \Rightarrow \quad dx_t = \bar{x}\hat{x}_t$$

Labor Supply Equation [1]

Consider a model with the same settings as in Question 2, with the following utility function:

$$\ln C_t + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1)$$

The household maximizes:

$$\max_{C_t, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) \right)$$

subject to the budget constraint:

$$C_t + K_{t+1} = w_t N_t + (1 + r_{k,t} - \delta) K_t + \Pi_t$$

where $L_t = 1 - N_t$ for an individual household.
First-order conditions (FOCs):

$$[C_t] \quad \frac{1}{C_t} = \lambda_t \tag{7}$$

$$[N_t] \quad \theta L_t^{-\eta} = w_t \lambda_t \tag{8}$$

[1] 2025 Midterm

3. From Linearization to Computation

Linearized DSGE Model: States and Controls

We write the linearized DSGE model as:

$$A\,E_t \begin{bmatrix} \hat{x}_{t+1} \\ \hat{w}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix}$$

where

$$\hat{x}_t = \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}, \qquad \hat{w}_t = \begin{bmatrix} \hat{c}_t \end{bmatrix}$$

- x_t: **predetermined (state) variables**
- w_t: **jump (control) variables**

From Linearized Equations to Matrix Form

Goal

Starting from the linearized equilibrium conditions, we want to rewrite the model as a linear system:

$$A\,\mathbb{E}_t[x_{t+1}] = B\,x_t$$

which can be solved using the Blanchard–Kahn method.

- Eliminate auxiliary variables (y_t, i_t, r_t)
- Express everything in terms of $(\hat{k}_t, \hat{a}_t, \hat{c}_t)$
- Stack equations into matrices A and B

Step 1-1: Steady-State Return

The gross return on capital is

$$r_{t+1} = \alpha a_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta$$

In the deterministic steady state:

$$a = 1, \quad k_{t+1} = k_t = k$$

Thus,

$$r = \alpha k^{\alpha-1} + 1 - \delta$$

From the steady-state Euler equation,

$$1 = \beta r \quad \Rightarrow \quad r = \frac{1}{\beta}$$

Therefore,

$$\alpha k^{\alpha-1} = \frac{1}{\beta} - (1 - \delta)$$

Step 1-2: Linearized Return

Linearizing

$$r_{t+1} = \alpha a_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta$$

around the steady state gives:

$$\hat{r}_{t+1} = \frac{\alpha k^{\alpha-1}}{\alpha k^{\alpha-1} + 1 - \delta}\left(\hat{a}_{t+1} - (1-\alpha)\hat{k}_{t+1}\right)$$

we obtain:

$$\hat{r}_{t+1} = \frac{\frac{1}{\beta} - (1-\delta)}{\frac{1}{\beta}}\left(\hat{a}_{t+1} - (1-\alpha)\hat{k}_{t+1}\right)$$

Step 1-3: Euler Equation Substitution

The linearized Euler equation is:

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] - \mathbb{E}_t[\hat{r}_{t+1}]$$

Substituting the expression for $\hat{r}_{t+1}$:

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] - \frac{\frac{1}{\beta} - (1-\delta)}{\frac{1}{\beta}} \mathbb{E}_t\left[\hat{a}_{t+1} - (1-\alpha)\hat{k}_{t+1}\right]$$

Rearranging terms:

$$\mathbb{E}_t[\hat{c}_{t+1}] + \frac{\frac{1}{\beta} - (1-\delta)}{\frac{1}{\beta}}(1-\alpha)\mathbb{E}_t[\hat{k}_{t+1}] - \frac{\frac{1}{\beta} - (1-\delta)}{\frac{1}{\beta}}\mathbb{E}_t[\hat{a}_{t+1}] = \hat{c}_t$$

Step 2-1: From Resource Constraint to Capital Dynamics

Start from steady state identities:

$$y = c + i, \qquad \frac{C}{Y} = 1 - \delta\frac{K}{Y}, \qquad \frac{I}{Y} = \delta\frac{K}{Y}$$

Linearizing the resource constraint:

$$\hat{y}_t = \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{i}_t = \frac{C}{Y}\hat{c}_t + \delta\frac{K}{Y}\hat{i}_t$$

From the production function:

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t$$

Equating the two expressions:

$$\delta\frac{K}{Y}\hat{i}_t = \hat{a}_t + \alpha\hat{k}_t - \frac{C}{Y}\hat{c}_t$$

Step 2-2: Capital Accumulation as a Matrix Equation

Linearized law of motion of capital:

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta \hat{i}_t$$

Substitute the expression for $\delta \hat{i}_t$:

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \frac{Y}{K}\left(\hat{a}_t + \alpha \hat{k}_t - \frac{C}{Y}\hat{c}_t\right)$$

Multiply both sides by $\frac{K}{Y}$:

$$\frac{K}{Y}\hat{k}_{t+1} = \hat{a}_t + \left[(1-\delta)\frac{K}{Y} + \alpha\right]\hat{k}_t - \frac{C}{Y}\hat{c}_t$$

Step 3: Final A and B Matrices

The linearized system can be written as:

$$\underbrace{\begin{bmatrix} (1-\alpha)\frac{R-(1-\delta)}{R} & -\frac{R-(1-\delta)}{R} & 1 \\ \frac{K}{Y} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \mathbb{E}_t \begin{bmatrix} \hat{k}_{t+1} \\ \hat{a}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ (1-\delta)\frac{K}{Y}+\alpha & 1 & -\frac{C}{Y} \\ 0 & \rho & 0 \end{bmatrix}}_{B} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \\ \hat{c}_t \end{bmatrix}$$

Interpretation

- Row 1: Euler equation (forward-looking)
- Row 2: Resource constraint + capital accumulation
- Row 3: Productivity shock process

Solution Structure from Blanchard–Kahn

After applying the Blanchard–Kahn conditions, the linearized system

$$A\mathbb{E}_t[x_{t+1}] = Bx_t$$

admits a unique stable solution of the form:

$$\begin{aligned} \hat{w}_t &= F\hat{x}_t \\ \hat{x}_{t+1} &= Q\hat{x}_t \end{aligned}$$

where the state–control vector is partitioned as:

$$\begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix} = \left[\begin{array}{c} \hat{k}_t \\ \hat{a}_t \\ \hline \hat{c}_t \end{array}\right]$$

Interpreting the Solution Matrices

The function `solab(A,B,nk)` returns two matrices:

$$\boxed{\hat{c}_t = F \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}} \qquad \boxed{\begin{bmatrix} \hat{k}_{t+1} \\ \hat{a}_{t+1} \end{bmatrix} = Q \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}}$$

- F: **policy function**
  - maps states into jump (control) variables
  - ensures forward-looking variables remain bounded
- Q: **law of motion of state variables**
  - governs the dynamic evolution of predetermined states
  - eigenvalues of Q are inside the unit circle

Optional) $1-\beta$ in Convergence Tolerance

Bellman operator is a contraction:

$$\|TV - TW\| \leq \beta \|V - W\|, \qquad 0 < \beta < 1$$

Since $V^* = TV^*$,

$$V - V^* = (V - TV) + (TV - TV^*)$$

Taking norms:

$$\|V - V^*\| \leq \|V - TV\| + \|TV - TV^*\|$$

Applying contraction:

$$\|V - V^*\| \leq \|V - TV\| + \beta \|V - V^*\|$$

$$\boxed{\|V - V^*\| \leq \frac{1}{1-\beta} \|TV - V\|}$$