# Graduate Macroeconomic Theory (ECN 200E)
# Discussion Section Week 2

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# Stochastic Growth Model

**Key Components:**

- Objective: Maximize lifetime utility

$$V(A, k) = \max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

  subject to:

$$c_t + k_{t+1} = A_t k_t^{\alpha}, \quad k_{t+1} \geq 0.$$

- Law of motion for productivity:

$$A_{t+1} \sim F(A_{t+1} | A_t).$$

# Euler Equation

The first-order condition for the optimization problem is:

$$u'(c_t) = \beta \mathbb{E}_t \left[ u'(c_{t+1}) \cdot \left( \alpha A_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right] .$$

The solution requires:

- A policy function $k_{t+1} = g(A_t, k_t)$,
- A value function $V(A, k)$ satisfying the Bellman equation.

## Guess and Verify: Policy Function

**Guess:** Policy function for capital:

$$k_{t+1} = \lambda A_t k_t^{\alpha}.$$

**Steps to Verify:**

1. Substitute the guess into the resource constraint:

$$c_t = (1-\lambda) A_t k_t^{\alpha}.$$

2. Compute marginal utility:

$$u'(c_t) = \frac{1}{(1-\lambda) A_t k_t^{\alpha}}.$$

3. Substitute into the Euler equation and simplify.
4. Solve for $\lambda$: $\lambda = \beta\alpha$.

**Result:** The policy function is:

$$k_{t+1} = \beta\alpha A_t k_t^{\alpha}.$$

# Guess and Verify: Value Function

**Guess:** Value function of the form:

$$V(A,k) = D + E\log(k) + F\log(A),$$

where $D, E, F$ are constants to be determined.

**Steps to Verify:**

1. Substitute the guess into the Bellman equation:

$$V(A,k) = \max_{k'} \left\{ u(c) + \beta \mathbb{E}\left[ V(A', k') \right] \right\}.$$

2. Compute:

$$c = Ak^{\alpha} - k', \quad u(c) = \log(c).$$

3. Derive the optimal $k'$ and substitute back into $V(A,k)$.

# Why This Functional Form?

- Preferences: $u(c) = \log c$
- Technology: $y = Ak^{\alpha}$
- Resource constraint implies $c$ is proportional to $Ak^{\alpha}$

**Key implication:**

$$\log c = \text{constant} + \log A + \alpha \log k$$

**Intuition:** Utility, production, and consumption are all log-linear in states.

# Why Does the Bellman Equation Close?

**Guess:**

$$V(A, k) = D + E \log k + F \log A$$

**Bellman equation:**

$$V(A, k) = \max_{k'} \left\{ \log c + \beta E[V(A', k')] \right\}$$

- $\log c$ is linear in $\log k$ and $\log A$
- $E[V(A', k')]$ preserves log-linearity
- RHS has the same functional form as the guess

**Result:** Only the coefficients $(D, E, F)$ need to be determined.

# Why Is This a Special Case?

- Requires log utility
- Requires Cobb–Douglas technology
- Requires simple capital accumulation
- Requires no constraints or nonlinearities

**If any of these break:**

- The Bellman equation no longer preserves the guessed form
- Additional nonlinear terms appear
- Guess and verify fails

**Conclusion:** Guess and verify works only in knife-edge cases.

# Conclusion

**Key Results:**

- **Policy Function:**

$$k_{t+1} = \beta \alpha A_t k_t^{\alpha}.$$

- **Value Function:**

$$V(A, k) = D + \frac{\alpha}{1 - \beta\alpha} \log(k) + \frac{1}{1 - \beta\alpha} \log(A).$$

These results verify the functional forms for the policy and value functions under the stochastic growth model.

# Taylor Expansion: The Basics

**Taylor Series Expansion**

For a function $f(x)$, the Taylor expansion around a point $x_0$ is:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

**Linear Approximation (First-Order)**

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

# Linearization Method

1. **Special Case: Multiplicative Equation (log-separable)**

$$y_t = k_t^{\alpha}(a_t n_t)^{1-\alpha}$$

## Linearization Method 1: Log-Linearization

**Step 1: Take logs and Subtract the steady state**

$$\ln y_t = \alpha \ln k_t + (1-\alpha)(\ln a_t + \ln n_t)$$

**Step 2: Express in deviations from steady state**

$$\ln \frac{y_t}{\bar{y}} = \alpha \ln \frac{k_t}{\bar{k}} + (1-\alpha)\left(\ln \frac{a_t}{\bar{a}} + \ln \frac{n_t}{\bar{n}}\right)$$

**Key approximation**

$$\ln \frac{x_t}{\bar{x}} = \ln\left(1 + \frac{x_t - \bar{x}}{\bar{x}}\right) \approx \frac{x_t - \bar{x}}{\bar{x}} \equiv \hat{x}_t$$

**Result (log-linear form)**

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)(\hat{a}_t + \hat{n}_t)$$

# Linearization Method 2: Total Differentiation

**Step 1: Total differentiation**

$$dy_t = \left.\frac{\partial y}{\partial k}\right|_{\bar{k},\bar{a},\bar{n}} dk_t + \left.\frac{\partial y}{\partial a}\right|_{\bar{k},\bar{a},\bar{n}} da_t + \left.\frac{\partial y}{\partial n}\right|_{\bar{k},\bar{a},\bar{n}} dn_t$$

$$dy_t = \alpha \bar{k}^{\alpha-1}(\bar{a}\bar{n})^{1-\alpha} dk_t + (1-\alpha)\bar{k}^{\alpha}\bar{a}^{-\alpha}\bar{n}^{1-\alpha} da_t + (1-\alpha)\bar{k}^{\alpha}\bar{a}^{1-\alpha}\bar{n}^{-\alpha} dn_t$$

**Key identity**

$$\frac{dx_t}{x_t} = \frac{x_t - \bar{x}}{\bar{x}} \equiv \hat{x}_t \quad \Rightarrow \quad dx_t = \bar{x}\hat{x}_t$$

**Step 2: Divide by $y_t$**

$$\frac{dy_t}{y_t} = \alpha \frac{dk_t}{k_t} + (1-\alpha)\left(\frac{da_t}{a_t} + \frac{dn_t}{n_t}\right)$$

# Linearization Method

**2. General Case**

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[ \frac{1}{c_{t+1}} r_{t+1} \right]$$
$$r_{t+1} = \alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)$$

# Linearizing Euler Equation (1/2)

The household Euler equation is:

$$\frac{1}{c_t} = \beta E_t \left[ \frac{1}{c_{t+1}} r_{t+1} \right]$$

Totally differentiate the Euler equation:

$$-\frac{1}{\bar{c}^2} dc_t = \beta E_t \left[ -\frac{\bar{r}}{\bar{c}^2} dc_{t+1} + \frac{1}{\bar{c}} dr_{t+1} \right] \tag{1}$$

$$-\hat{c} = \beta r E_t \left[ -\hat{c}_{t+1} + \hat{r}_{t+1} \right] \tag{2}$$

At the steady state:

$$\frac{1}{\bar{c}} = \beta \frac{1}{\bar{c}} \bar{r} \quad \Rightarrow \quad \beta \bar{r} = 1$$

# Linearizing Euler Equation (2/2)

Rearranging terms:

$$\hat{c}_t = E_t \hat{c}_{t+1} - E_t \hat{r}_{t+1}$$

**Interpretation:**

- Consumption is forward-looking
- Expected returns shift consumption across time

# Linearizing Return on Capital (1/2)

The gross return on capital is:

$$r_{t+1} = \alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)$$

Totally differentiate:

$$dr_{t+1} = \alpha(\alpha-1)\bar{a}\bar{k}^{\alpha-2} dk_{t+1} + \alpha \bar{k}^{\alpha-1} da_{t+1}$$

Divide by $\bar{r}$:

$$\hat{r}_{t+1} = \frac{\alpha \bar{a}\bar{k}^{\alpha-1}}{\bar{r}} \left[\hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1}\right]$$

# Linearizing Return on Capital (2/2)

At the steady state:

$$\bar{r} = \bar{\alpha}\bar{k}^{\alpha-1} + (1-\delta)$$

$$\therefore \hat{r}_{t+1} = \frac{\alpha \bar{a}\bar{k}^{\alpha-1}}{\bar{\alpha}\bar{k}^{\alpha-1} + (1-\delta)} \left[\hat{a}_{t+1} - (1-\alpha)\hat{k}_{t+1}\right]$$

# Linearization Example [1]

**Labor Supply Equations:**

- When $\epsilon = 0$:

$$\eta N_t^{\eta-1} = w_t$$

- When $\epsilon = 1$:

$$\eta N_t^{\eta-1} \frac{1}{(1-N_t)\lambda_t} = w_t,$$

where $\frac{1}{C_t} = \lambda_t$ is the marginal utility of consumption.

**Question: Linearize the two labor supply conditions above.**

[1] 2021 Midterm Q3

# Code example: What Model Are We Solving?

We solve a **stochastic growth model** with full depreciation.

**Preferences:**

$$\max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \log c_t$$

**Technology:**

$$y_t = z_t A k_t^{\alpha}$$

**Resource constraint:**

$$c_t + k_{t+1} = z_t A k_t^{\alpha}$$