Graduate Macroeconomic Theory (ECN 200E)
Discussion Section Week 1

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Logistics

- **Discussion section**: Friday, 9:00–9:50 AM, Blue Room
- **Office hours**: Friday, 2:00–4:00 PM, Dutton 3132
  - *Tentative reschedule*: Jan 23 (F) $\rightarrow$ Jan 21 (W), 2:00–4:00 PM
- **Homework submission**: jnhkim@ucdavis.edu

Recap of Deterministic Growth Model

Key Equations:

$$V^*(k_0) = \max_{k_{t+1}, c_t} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Subject to:

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + i_t && \text{(Capital Accumulation)} \\ y_t &= f(k_t) && \text{(Production Function)} \\ y_t &= c_t + i_t && \text{(Resource Constraint)} \end{aligned}$$

Assumptions:

- Deterministic environment (no uncertainty or shocks).
- Recursive versus sequential setups.

Dynamic Programming Problem

The value function is defined as:

$$V(k_t) = \max_{k_{t+1}} [\ln(c_t) + \beta V(k_{t+1})]$$

Substitute the resource constraint:

$$c_t = Ak_t^{\alpha} + (1-\delta)k_t - k_{t+1}$$

The Bellman equation becomes:

$$V(k_t) = \max_{k_{t+1}} [\ln(Ak_t^{\alpha} + (1-\delta)k_t - k_{t+1}) + \beta V(k_{t+1})]$$

Key Concepts: State and Control Variables

State Variables

- Variables that are known and realized by agents before they make decisions in the current period.
- Evolve gradually over time, often reflecting accumulated effects of past decisions or shocks.
- Can be classified as:
  - Endogenous (e.g., K_t): Determined within the model.
  - Exogenous (e.g., ϵ_t): Determined outside the model.

Control Variables

- Variables that decision-makers actively choose each period based on the current state.
- Can change freely within the period and may exhibit abrupt adjustments or "jumps."

Example: Guess and Verify

Model Specifications:

- Utility: $u(c_t) = \ln(c_t)$
- Production: $f(k_t) = Ak_t^\alpha$
- Full depreciation: $\delta = 1$, so $k_{t+1} = i_t$

Guess:

$$k_{t+1} = \phi A k_t^\alpha$$

Substitute the guess into the resource constraint:

$$c_t + \phi A k_t^\alpha = A k_t^\alpha$$

Solve for c_t:

$$c_t = (1 - \phi) A k_t^\alpha$$

Example: Guess and Verify

The Bellman equation is:

$$V(k_t) = \max_{k_{t+1}} [\ln(c_t) + \beta V(k_{t+1})]$$

The FOC is:

$$\frac{-1}{c_t} + \beta V'(k_{t+1}) = 0$$

By envelope theorem, we get

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} A\alpha k_{t+1}^{\alpha-1}$$

After a bit of algebra,

$$\phi = \alpha\beta$$

Then express control variables as a function of state variables.

Introducing Stochastic Growth

Goal: Model economies that grow and fluctuate.

- Incorporates uncertainty via stochastic TFP a_t.
- Adds a state variable a_t, governed by a stochastic process.
- Example: $y_t = a_t f(k_t)$.

Markov Chains - Basics

Definition:

$$P(s_{t+1}|s_t, s_{t-1}, \ldots) = P(s_{t+1}|s_t)$$

Components:

- **State Space:** $S = \{s_1, s_2, \ldots, s_n\}$
- **Transition Matrix:** $P_{ij} = P(s_{t+1} = s_j | s_t = s_i)$
- **Initial Distribution:** π_0

Markov Chains - Economic Example

Example: GDP Growth

- States: Boom (s_1) and Recession (s_2).
- Outcomes: $y = [1.2, -0.4]'$.

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix}$$

Interpretation:

- Booms are persistent with a 90% probability.
- Recessions have a higher chance of transitioning to booms.

Stationary Distribution

Evolution of Probabilities:

$$\pi'_{t+1} = \pi'_t P \quad \text{and} \quad \pi'_k = \pi'_0 P^k$$

Stationary Distribution:

$$\pi' = \pi' P$$

- Explains long-run behavior of the Markov chain.
- Found as the eigenvector of P' associated with eigenvalue 1.

Contraction Mapping Theorem

Why It's Important:

- Ensures the Bellman equation has a unique solution.
- Guarantees convergence of the value function iteration.

Theorem:

- If T is a contraction mapping on a complete metric space, there exists a unique fixed point x^* such that $T(x^*) = x^*$.

Examples for Markov Processes

Markov Chain vs. Markov Process:

- Chain: Discrete state space (e.g., boom/recession).
- Process: Continuous state space (e.g., $\ln a_{t+1} = \rho \ln a_t + \epsilon_t$).

Discretization Methods:

- Tauchen or Rouwenhorst method for finite-state approximations.

Why discretization?

- The AR(1) process has a continuous state space
- Solving Bellman equations numerically requires finite states
- Discretization turns integrals into matrix multiplications

Definition of Ergodic Sets

Ergodic Set: A subset of states C in a Markov chain that is both:

- **Closed:** Once the process enters C, it cannot leave C.
- **Irreducible:** Every state in C communicates with every other state in C.

Key Points:

- The entire state space of an irreducible Markov chain is an ergodic set.
- Ergodic sets help identify long-run equilibrium behaviors in Markov chains.
- Ergodic set is the support of the stationary distribution

Intuition:

- The set of states that are *visited repeatedly in the long run*
- The region where the stationary distribution puts its mass

When Ergodic Sets Matter

In standard DSGE models

- Finite state space with an irreducible and aperiodic transition matrix
- Entire state space forms a single ergodic set

  $\Rightarrow$ a unique stationary distribution exists

Ergodic sets become important when:

- The state space is no longer fully connected
- Some states are absorbing or nearly absorbing
- Transition probabilities depend on the state or policy regime

Examples:

- Zero Lower Bound (ZLB)
- Occasionally Binding Constraints (OccBin)
- Default, bankruptcy, or unemployment traps

Example 1: Simple Two-State Markov Chain

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

State Space: $S = \{1, 2\}$

- Irreducible: Both states communicate ($P(1 \to 2) > 0$, $P(2 \to 1) > 0$).
- Closed: There are no transitions outside S.

Ergodic Set: $S = \{1, 2\}$

Example 2: Absorbing State

$$P = \begin{bmatrix} 0.5 & 0.5 & 0.0 \\ 0.2 & 0.5 & 0.3 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

State Space: $S = \{1, 2, 3\}$

- $C_1 = \{3\}$: Closed and irreducible (absorbing state).
- $C_2 = \{1, 2\}$: It is not clsed. From State 2, the chain can transition to State 3. Once in State 3, the chain cannot return to States 1 or 2.

Ergodic Sets:

- $C_1 = \{3\}$

Example 3: Two Disjoint Cycles

$$P = \begin{bmatrix} 0.5 & 0.5 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.7 & 0.3 \\ 0.0 & 0.0 & 0.6 & 0.4 \end{bmatrix}$$

State Space: $S = \{1, 2, 3, 4\}$

- $C_1 = \{1, 2\}$: Closed and irreducible.
- $C_2 = \{3, 4\}$: Closed and irreducible.

Ergodic Sets:

- $C_1 = \{1, 2\}$
- $C_2 = \{3, 4\}$

Example 4: Weather Example

States:

- $S = \{1 : \text{Sunny}, 2 : \text{Cloudy}, 3 : \text{Rainy}\}$

$$P = \begin{bmatrix} 0.8 & 0.2 & 0.0 \\ 0.3 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Analysis:

- Irreducible: Every state communicates with every other state.
- Closed: No transitions outside S.

Ergodic Sets: $C = \{1, 2, 3\}$

Example 5: Identity Matrix

$$P = I_3$$

State Space: $S = \{1, 2, 3\}$

- Each state is absorbing (no transitions to other states).
- No communication between states.

Ergodic Sets: $\{1\}, \{2\}, \{3\}$

Summary

- **Deterministic vs. stochastic models**: uncertainty implies long-run distributions, not point convergence.
- **Markov structure**: the current state summarizes all relevant past information.
- **Stationary distribution**: describes long-run behavior of the stochastic economy.
- **Contraction mapping theorem**: ensures existence, uniqueness, and convergence of solutions.